# 2.

# Polar form of a complex number

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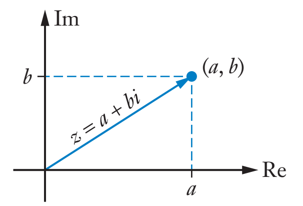
# Polar form of a complex number

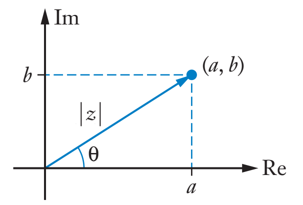
Modulus and argument

The *Preliminary work* section at the beginning of this book reminded you of the idea of representing a complex number $a + bi$ graphically on an **Argand diagram**, either as a point $(a, b)$ or as a vector from the origin to the point $(a, b)$.




If we write $|z|$ for the magnitude of this vector, and if it makes an angle $\theta$ with the positive real axis, see diagram, then:

$$\begin{aligned} z &= a + bi \\ &= |z|\cos\theta + i|z|\sin\theta \\ &= |z|(\cos\theta + i\sin\theta) \end{aligned}$$




This is the **polar form** of the complex number $z$.

(Some calculators may refer to this as the '*Trig*' form.)

Note

- The magnitude of the complex number, $|z|$, is called the **modulus** of $z$, written mod $z$. (The letter $r$ is also used for the modulus of a complex number.)
- With $z = a + bi$, $|z| = \sqrt{a^2 + b^2}$.
  $|z| = \sqrt{a^2 + b^2}$ is the distance the point representing $a + bi$ is from the origin.
  It is the magnitude of the vector representing $a + bi$.
- The angle $\theta$, measured anticlockwise from the positive real axis, is said to be the **argument** of the complex number, written arg $z$.
  Arg $z$ is usually stated in radians but can be given in degrees.
  Arg $z$ is not defined for the number $(0 + 0i)$.
- With $\theta$ in radians we could also refer to $(\theta \pm 2\pi)$, $(\theta \pm 4\pi)$, $(\theta \pm 6\pi)$, etc. as being the argument of a complex number that makes an angle $\theta$ with the positive real axis. To avoid this confusion we refer to the value $-\pi < \theta \leq \pi$ (or in degrees $-180° < \theta \leq 180°$) as the **principal argument** of the complex number. The interval $-\pi < \theta \leq \pi$ is sometimes written as $(-\pi, \pi]$. In this form, note carefully the significance of the type of bracket used.

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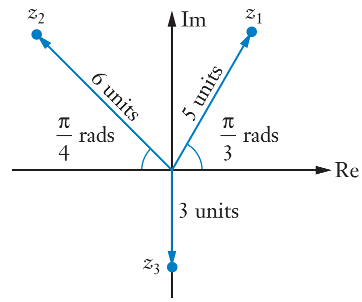
For the Argand diagram on the right:

$z_1$ has modulus 5 and principal argument $\dfrac{\pi}{3}$.

$z_2$ has modulus 6 and principal argument $\dfrac{3\pi}{4}$.

$z_3$ has modulus 3 and principal argument $-\dfrac{\pi}{2}$.

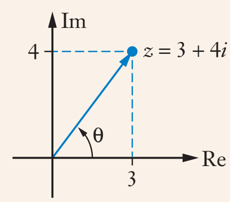



## EXAMPLE 1

Express the complex number $3 + 4i$ in the form $r(\cos\theta + i\sin\theta)$, for $-\pi < \theta \le \pi$, and with $\theta$ given correct to four decimal places.

### Solution

From the sketch on the right we see that




$$\begin{aligned} \text{mod } z &= \sqrt{3^2 + 4^2} \\ &= 5 \end{aligned}$$

and $\quad \tan\theta = \dfrac{4}{3}$

$\therefore \quad \arg z = 0.9273 \text{ rads} \quad$ (correct to 4 decimal places)

Thus $\quad z = 5(\cos 0.9273 + i\sin 0.9273)$

Alternatively we could use the ability of some calculators to

- determine mod $z$ and arg $z$,
- to convert cartesian coordinates to polar coordinates direct.

$|3 + 4i|$ → 5

$\arg(3 + 4i)$ → 0.927295218

$\text{toPol}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$ → $\begin{bmatrix} 5 \\ \angle(0.927295218) \end{bmatrix}$

Thus, as before, the complex number

$z = 3 + 4i$

has the **polar form**:

$z = 5(\cos 0.9273 + i\sin 0.9273)$

Note

- $z = a + ib$ is the **rectangular** or **cartesian** form of the complex number $z$.

  This cartesian form is sometimes written as the ordered pair $(a, b)$.

- In the polar form, $z = r(\cos\theta + i\sin\theta)$, $r$ and $\theta$ are real and it is usual to have

  $$r \ge 0 \text{ and } -\pi < \theta \le \pi.$$

  This polar form can also be written as an ordered pair $(r, \theta)$, with squared brackets $[r, \theta]$ sometimes used to distinguish polar form.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## Exercise 2A

**1** Find $|z|$ for each of the following, giving exact answers.

**a** $z = 4 - 3i$

**b** $z = 12 + 5i$

**c** $z = 3 + 2i$

**d** $z = 3 - 2i$

**e** $z = 1 + 5i$

**f** $z = 5i$

**2** Find the principal argument of each of the following complex numbers, giving exact answers in radians.

**a** $z = 2 + 2i$

**b** $z = 2 - 2i$

**c** $z = -2 + 2i$

**d** $z = -2 - 2i$

**e** $z = -2 + 2\sqrt{3}i$

**f** $z = 3 - 3\sqrt{3}i$

**3** Express the complex numbers $z_1$ to $z_{12}$, given below, in the form

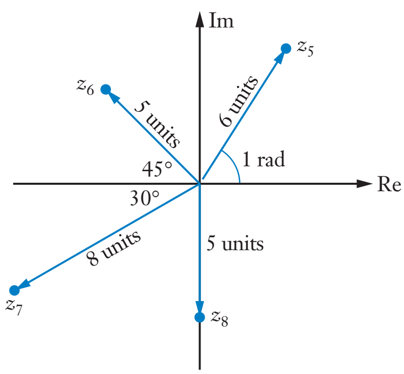
$$r(\cos\theta + i\sin\theta), \text{ with } r \geq 0 \text{ and } -\pi < \theta \leq \pi.$$

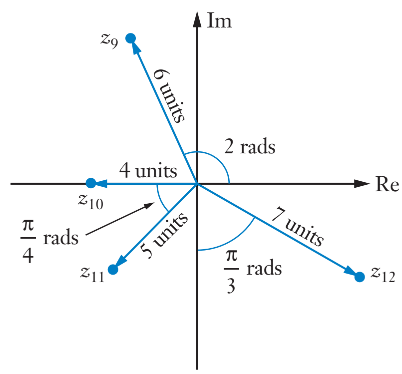
$z_1 = 3\left(\cos\dfrac{13\pi}{6} + i\sin\dfrac{13\pi}{6}\right)$

$z_2 = 3(\cos 3\pi + i\sin 3\pi)$

$z_3 = 4\left(\cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4}\right)$

$z_4 = 2(\cos(-\pi) + i\sin(-\pi))$

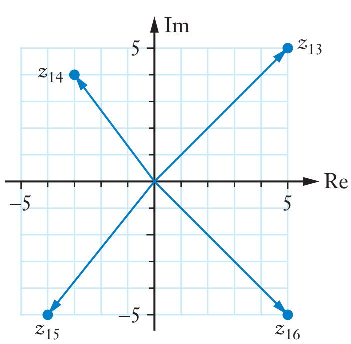



**4** Express the complex numbers $z_{13}$ to $z_{26}$, given below, in the form

$$r(\cos\theta + i\sin\theta), \text{ with } r \geq 0 \text{ and } -\pi < \theta \leq \pi$$

stating $r$ exactly and $\theta$ correct to 4 decimal places (if rounding is necessary).




$z_{17} = 5 + 12i$

$z_{18} = 1 + 7i$

$z_{19} = 1 - 7i$

$z_{20} = -7 + i$

$z_{21} = 5\sqrt{3} + 5i$

$z_{22} = 4i$

$z_{23} = 4$

$z_{24} = -4$

$z_{25} = -3i$

$z_{26} = 3$

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**5** Express $z_{27}$ to $z_{32}$ in the form $a + bi$, with $a$ and $b$ stated exactly.

$$z_{27} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \qquad z_{28} = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

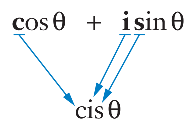
$$z_{29} = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) \qquad z_{30} = 6\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

$$z_{31} = 5(\cos 2\pi + i\sin 2\pi) \qquad z_{32} = \left(\cos\frac{7\pi}{2} + i\sin\frac{7\pi}{2}\right)$$

WS Complex number conversions

## The abbreviated form for $\cos\theta + i\sin\theta$

To save us having to write $\cos\theta + i\sin\theta$ we can use the abbreviation $\text{cis}\,\theta$.




$$\begin{aligned}
\text{Thus} \qquad r(\cos\theta + i\sin\theta) &= r\,\text{cis}\,\theta. \\
\text{e.g.} \qquad 5(\cos 2 + i\sin 2) &= 5\,\text{cis}\,2. \\
4\,\text{cis}\,\frac{\pi}{3} &= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\
&= 2 + 2\sqrt{3}i.
\end{aligned}$$

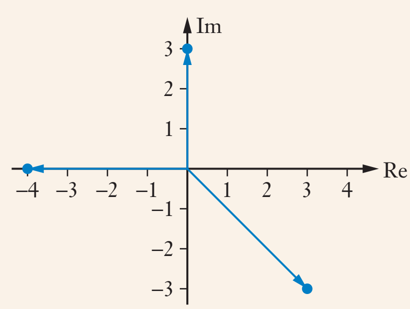
**EXAMPLE 2**

Express each of the following in the form $r\,\text{cis}\,\theta$, with $r \geq 0$ and $-\pi < \theta \leq \pi$.

**a** $3 - 3i$, **b** $3i$, **c** $-4$.

**Solution**

**a**
$$\begin{aligned}
\text{mod}(3 - 3i) &= \sqrt{3^2 + (-3)^2} \\
&= 3\sqrt{2} \\
\arg(3 - 3i) &= -\frac{\pi}{4} \\
\therefore \quad 3 - 3i &= 3\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)
\end{aligned}$$




**b** $3i = 3\,\text{cis}\,\frac{\pi}{2}$

**c** $-4 = 4\,\text{cis}\,\pi$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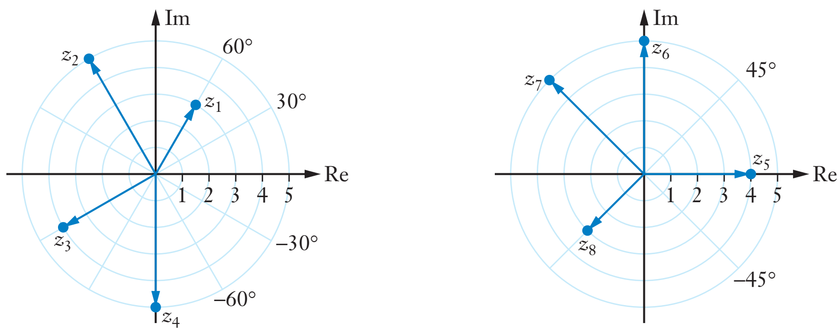
## Exercise 2B

**1** Express $z_1$ to $z_8$ shown below in the form $r\operatorname{cis}\theta$, with $r \geq 0$ and $-\pi < \theta \leq \pi$.




Express each of the following in the form $r\operatorname{cis}\theta$, with $r \geq 0$ and $-\pi < \theta \leq \pi$.

**2** $2\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

**3** $7\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)$

**4** $9(\cos 30° + i\sin 30°)$

**5** $3(\cos 330° + i\sin 330°)$

**6** $5\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

**7** $4\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)$

**8** $2\left(\cos\left(-\frac{5\pi}{3}\right) + i\sin\left(-\frac{5\pi}{3}\right)\right)$

**9** $2(\cos(-3\pi) + i\sin(-3\pi))$

Simplify:

**10** $7\operatorname{cis}\frac{\pi}{2}$

**11** $5\operatorname{cis}\left(-\frac{\pi}{2}\right)$

**12** $\operatorname{cis}\pi$

**13** $3\operatorname{cis}2\pi$

Express each of the following in the form $a + bi$, with exact values for $a$ and $b$.

**14** $10\operatorname{cis}\frac{\pi}{4}$

**15** $4\operatorname{cis}\frac{2\pi}{3}$

**16** $4\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

**17** $12\operatorname{cis}\left(-\frac{4\pi}{3}\right)$

Express each of the following in the form $r\operatorname{cis}\theta$, for $-\pi < \theta \leq \pi$. Give $r$ ($\geq 0$) as an exact value and, if rounding is necessary, give $\theta$ correct to 4 decimal places.

**18** $-7 + 24i$

**19** $-5 + 12i$

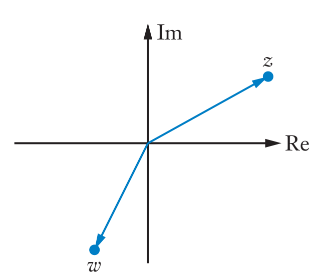
**20** $1 + 2i$

**21** $5i$

**22** If $z = a + bi$ we define $\bar{z}$, the complex **conjugate** of $z$, as $\bar{z} = a - bi$.

**a** Make a copy of the Argand diagram shown on the right and include $\bar{z}$ and $\bar{w}$.

**b** If $z = r_1\operatorname{cis}\alpha, -\pi < \alpha \leq \pi$,
and $w = r_2\operatorname{cis}\beta, -\pi < \beta \leq \pi$,
express $\bar{z}$ and $\bar{w}$ in 'cis form'.

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Find the complex conjugates of the following, giving your answers in polar form, $r \operatorname{cis} \theta°$, with $r \geq 0$ and $-180 < \theta \leq 180$.

**23** $2 \operatorname{cis} 30°$

**24** $7 \operatorname{cis} 120°$

**25** $4 \operatorname{cis} 390°$

**26** $10 \operatorname{cis} (-200°)$

Find the complex conjugates of the following, giving answers in polar form, $r \operatorname{cis} \theta$, with $r \geq 0$ and $-\pi < \theta \leq \pi$.

**27** $2 \operatorname{cis} \frac{\pi}{2}$

**28** $5 \operatorname{cis} \left(-\frac{3\pi}{4}\right)$

**29** $5 \operatorname{cis} 0.5$

**30** $5 \operatorname{cis} \frac{7\pi}{2}$

WS

Polar complex number operations

## Multiplying and dividing complex numbers expressed in polar form

Suppose that $z = r_1 \operatorname{cis} \alpha = r_1(\cos \alpha + i \sin \alpha)$ and $w = r_2 \operatorname{cis} \beta = r_2(\cos \beta + i \sin \beta)$

Then

$$\begin{aligned} zw &= (r_1 \operatorname{cis} \alpha)(r_2 \operatorname{cis} \beta) \\ &= r_1 r_2(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \\ &= r_1 r_2[(\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)] \\ &= r_1 r_2[\cos (\alpha + \beta) + i \sin (\alpha + \beta)] \\ &= r_1 r_2 \operatorname{cis} (\alpha + \beta) \end{aligned}$$

Or, using the 'square bracket' notation: $[r_1, \alpha][r_2, \beta] = [r_1 r_2, \alpha + \beta]$

Thus: **When we multiply two complex numbers we multiply the moduli and add the arguments** (adding or subtracting multiples of $2\pi$ to ensure $-\pi < \arg \leq \pi$).

The reader is left to confirm that if $z = r_1 \operatorname{cis} \alpha$ and $w = r_2 \operatorname{cis} \beta$

then $\frac{z}{w} = \frac{r_1}{r_2} \operatorname{cis} (\alpha - \beta)$.

I.e: **When we divide two complex numbers we divide the moduli and subtract the arguments** (adding or subtracting multiples of $2\pi$ to ensure $-\pi < \arg \leq \pi$).

(Question: Is the order of subtraction of the angles important?)


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**EXAMPLE 3**

If $z = 5\operatorname{cis}\frac{3\pi}{4}$ and $w = 2\operatorname{cis}\frac{\pi}{3}$ express each of the following in the form $r\operatorname{cis}\theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$.

**a** $zw$ **b** $\frac{z}{w}$ **c** $2z$ **d** $iz$

**Solution**

**a**
$$\begin{aligned} zw &= 5\operatorname{cis}\frac{3\pi}{4} \cdot 2\operatorname{cis}\frac{\pi}{3} \\ &= 10\operatorname{cis}\left(\frac{3\pi}{4} + \frac{\pi}{3}\right) \\ &= 10\operatorname{cis}\frac{13\pi}{12} \\ &= 10\operatorname{cis}\left(-\frac{11\pi}{12}\right) \end{aligned}$$

**b**
$$\begin{aligned} \frac{z}{w} &= \left(5\operatorname{cis}\frac{3\pi}{4}\right) \div \left(2\operatorname{cis}\frac{\pi}{3}\right) \\ &= \frac{5}{2}\operatorname{cis}\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) \\ &= 2.5\operatorname{cis}\frac{5\pi}{12} \end{aligned}$$

**c**
$$\begin{aligned} 2z &= 2\operatorname{cis}0 \cdot 5\operatorname{cis}\frac{3\pi}{4} \\ &= 10\operatorname{cis}\frac{3\pi}{4} \end{aligned}$$

**d**
$$\begin{aligned} iz &= 1\operatorname{cis}\frac{\pi}{2} \cdot 5\operatorname{cis}\frac{3\pi}{4} \\ &= 5\operatorname{cis}\left(-\frac{3\pi}{4}\right) \end{aligned}$$

## Geometrical interpretation

WS Multiplication in the plane

WS Division in the plane

If $z = r_1\operatorname{cis}\alpha$ and $w = r_2\operatorname{cis}\beta$

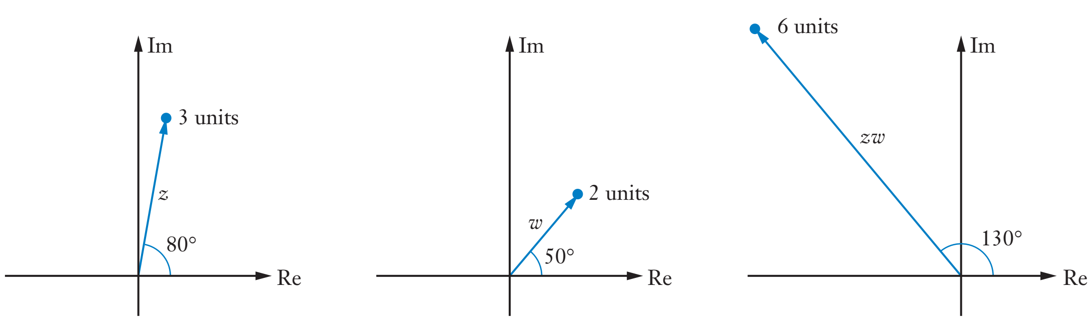
then $wz = r_1 r_2 \operatorname{cis}(\alpha + \beta)$.

Thus the effect of multiplying $z$ by $w$ is to rotate $z$ anticlockwise about the origin by an angle $\beta$, and increase the length of $z$ by a factor $|w|$.

For example suppose that $z = 3\operatorname{cis}80°$ and $w = 2\operatorname{cis}50°$.

It follows that
$$\begin{aligned} zw &= (3\operatorname{cis}80°)(2\operatorname{cis}50°) \\ &= 6\operatorname{cis}130°, \end{aligned}$$

i.e. a 50° anticlockwise rotation of $z$ about the origin with a doubling of length, as shown below.

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## Exercise 2C

Determine $zw$ for each of the following, giving your answer in the same form as $z$ and $w$ are given.

**1** $z = 2 + 3i, \quad w = 5 - 2i.$

**2** $z = 3 + 2i, \quad w = -1 + 2i.$

**3** $z = 3\operatorname{cis}60°, \quad w = 5\operatorname{cis}20°.$

**4** $z = 3\operatorname{cis}120°, \quad w = 3\operatorname{cis}150°.$

**5** $z = 3\operatorname{cis}30°, \quad w = 3\operatorname{cis}(-80°).$

**6** $z = 5\operatorname{cis}\dfrac{\pi}{3}, \quad w = 2\operatorname{cis}\dfrac{\pi}{4}.$

**7** $z = 4\operatorname{cis}\left(\dfrac{\pi}{4}\right), \quad w = 2\operatorname{cis}\left(-\dfrac{3\pi}{4}\right).$

**8** $z = 2\,(\cos 50° + i\sin 50°), \quad w = \cos 60° + i\sin 60°.$

**9** $z = 2(\cos 170° + i\sin 170°), \quad w = 3(\cos 150° + i\sin 150°).$

Determine $\dfrac{z}{w}$ for each of the following, giving your answer in the same form as $z$ and $w$ are given.

**10** $z = 6 - 3i, \quad w = 3 - 4i.$

**11** $z = -6 + 3i, \quad w = -3 + 4i.$

**12** $z = 8\operatorname{cis}60°, \quad w = 2\operatorname{cis}40°.$

**13** $z = 5\operatorname{cis}120°, \quad w = \operatorname{cis}150°.$

**14** $z = 3\operatorname{cis}(-150°), \quad w = 3\operatorname{cis}80°.$

**15** $z = 2\operatorname{cis}\dfrac{3\pi}{5}, \quad w = 2\operatorname{cis}\dfrac{2\pi}{5}.$

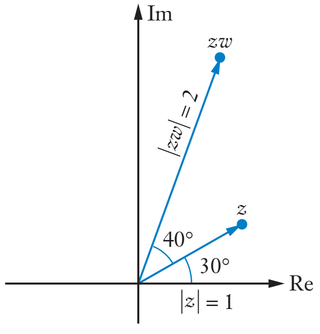
**16** $z = 4\operatorname{cis}\left(\dfrac{\pi}{4}\right), \quad w = 2\operatorname{cis}\left(-\dfrac{3\pi}{4}\right).$

**17** $z = 5\left(\cos\dfrac{3\pi}{4} + i\sin\left(\dfrac{3\pi}{4}\right)\right), \quad w = 2\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right).$

**18** $z = 2(\cos 50° + i\sin 50°), \quad w = 5(\cos 50° + i\sin 50°).$

Use the following diagrams to determine the complex number $w$ in the form $r\operatorname{cis}\theta$ for $r \geq 0$ and with $-180° < \theta \leq 180°$.

**19**

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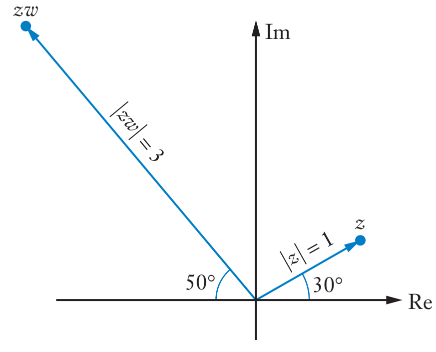



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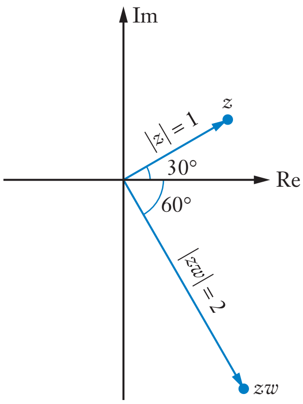


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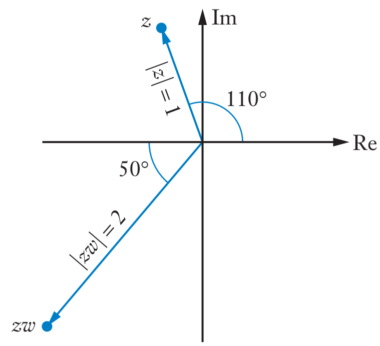


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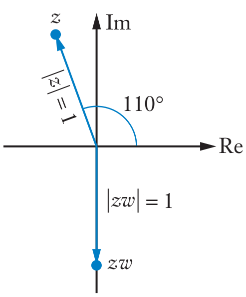


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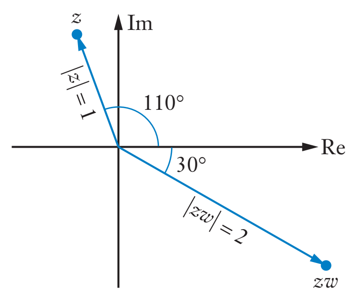


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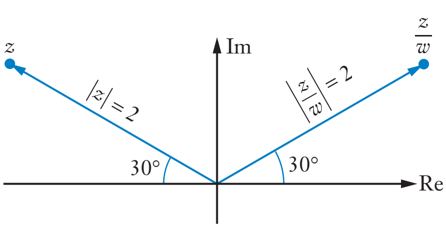


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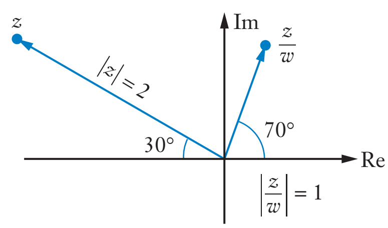


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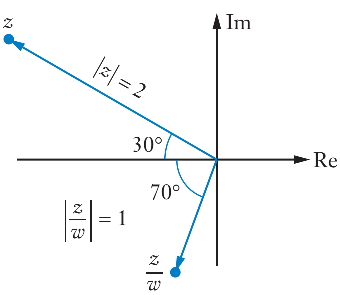


**28** If $z = 6\operatorname{cis} 40°$ and $w = 2\operatorname{cis} 30°$ determine

**a** $2z$ **b** $3w$ **c** $zw$ **d** $wz$

**e** $iz$ **f** $iw$ **g** $\dfrac{w}{z}$ **h** $\dfrac{1}{z}$

**29** If $z = 8\operatorname{cis}\dfrac{2\pi}{3}$ and $w = 4\operatorname{cis}\dfrac{3\pi}{4}$ determine

**a** $zw$ **b** $wz$ **c** $\dfrac{w}{z}$ **d** $\dfrac{z}{w}$

**e** $\overline{z}$ **f** $\overline{w}$ **g** $\dfrac{1}{z}$ **h** $\dfrac{i}{w}$

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Complex plane graphs

# Regions in the complex plane

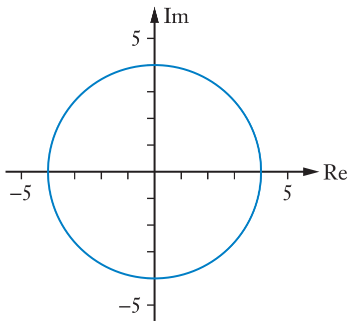
We know that we can represent the complex number $z = a + ib$ as a point $(a, b)$ on an Argand diagram. If we are then given some condition or rule that $z$ must obey, the set of all such $z$'s obeying the rule will form a set of points on our Argand diagram. For example, any complex number, $z$, obeying the rule

$$|z| = k$$

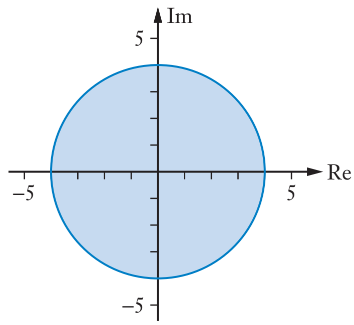
will be a distance of $k$ from the origin. The set of all complex numbers, $z$, obeying this rule, written $\{z: |z| = k\}$ and read as '*the set of all z such that mod z equals k*' would together form a circle, centre at the origin and radius $k$.

We say that the **locus** of $|z| = k$ is a circle centre the origin and radius $k$.

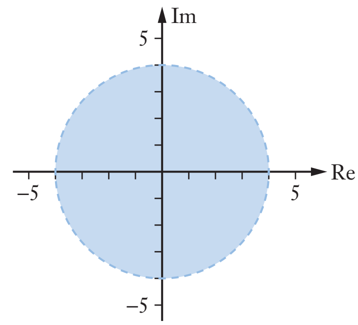
Examples:




$\{z: |z| = 4\}$



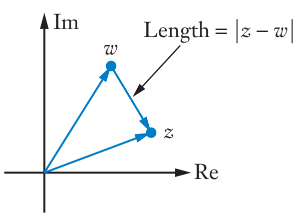

$\{z: |z| \leq 4\}$



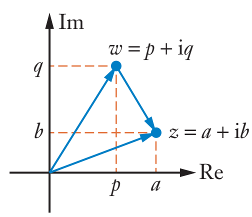

$\{z: |z| < 4\}$

Now consider $\{z: |z - w| = k\}$. The *Preliminary work* section at the beginning of this book reminded us that $|x - a|$ is the distance the number $x$ is from the number $a$. Similarly, for complex numbers $z$ and $w$, $|z - w|$ is the distance between $z$ and $w$ in the complex plane (see the diagrams below).

**Vector justification**




**Coordinate justification**




$$z - w = (a - p) + (b - q)i$$

$$|z - w| = \sqrt{(a - p)^2 + (b - q)^2}$$

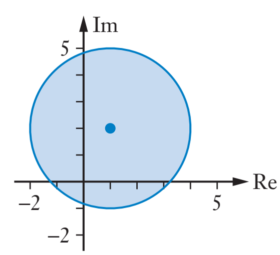
$$= \text{distance between } z \text{ and } w.$$

Thus the complex numbers, $z$, for which $|z - w| = k$, would together form a circle, centre at $w$ and radius $k$. We say that the locus of $|z - w| = k$ is a circle centre $w$ and radius $k$.

For example, the Argand diagram on the right shows the locus of all points in the complex plane for which

$$|z - (1 + 2i)| \leq 3.$$

i.e. it shows the set $\{z: |z - (1 + 2i)| \leq 3\}$





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## EXAMPLE 4

**a** Represent $\{z: |z - 2 - i| = 3\}$ on an Argand diagram.

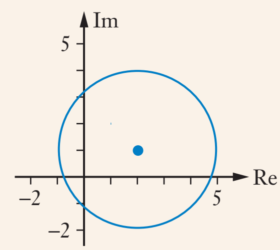
**b** If $z = x + iy$ determine the cartesian equation for the set of points in **a**.

### Solution

**a** $\{z: |z - w| = k\}$ is a circle centre $w$ and radius $k$.

Thus $\{z: |z - (2 + i)| = 3\}$ is a circle centre $(2 + i)$ and radius 3.

This is shown in the diagram on the right.




**b** Now $|z - 2 - i| = 3$

Thus if $z = x + iy$

$$|x + iy - 2 - i| = 3$$
$$|(x - 2) + i(y - 1)| = 3$$
$$\therefore \quad (x - 2)^2 + (y - 1)^2 = 9$$

Note: As we were reminded in the *Preliminary work* section at the beginning of this book, cartesian equations of the form

$$(x - p)^2 + (y - q)^2 = a^2$$

are those of circles, centre $(p, q)$ and radius $a$. Hence our answer for part **b**, above, confirms our answer for part **a**.

## EXAMPLE 5

**a** Represent $\{z: |z - 2i| = |z - 4|\}$ on an Argand diagram.

**b** If $z = x + iy$ determine the cartesian equation for the set of points in **a**.

### Solution

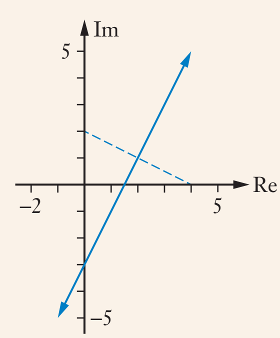
**a** $|z - w|$ is the distance from $z$ to $w$.

Thus $|z - 2i|$ is the distance from $z$ to $0 + 2i$,

and $|z - 4|$ is the distance from $z$ to $4 + 0i$.

Thus $\{z: |z - 2i| = |z - 4|\}$ is the set of all points equidistant from $2i$ and 4. It is the perpendicular bisector of the line joining $2i$ and 4.

This is shown in the diagram on the right.




**b** Now $|z - 2i| = |z - 4|$

Thus if $z = x + iy$

$$|x + iy - 2i| = |x + iy - 4|$$
$$x^2 + (y - 2)^2 = (x - 4)^2 + y^2$$

Which simplifies to $y = 2x - 3$

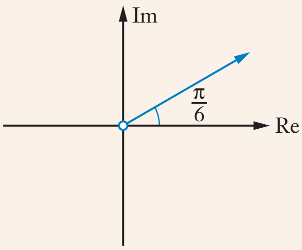
The required cartesian equation is $y = 2x - 3$

## EXAMPLE 6

Represent $\left\{z: \arg z = \frac{\pi}{6}\right\}$ on an Argand diagram.

### Solution

If $\arg z = \frac{\pi}{6}$ then $z$ makes an angle of $\frac{\pi}{6}$ with the positive $x$-axis, measured anticlockwise. Thus $\left\{z: \arg z = \frac{\pi}{6}\right\}$ is the set of points forming the 'half line' shown in the diagram on the right.




Note that the point $0 + 0i$ is not included because $\arg(0 + 0i)$ is undefined.

## EXAMPLE 7

Show that the set of all points $z$ in the complex plane that are such that

$$|z - (8 + i)| = 2|z - (2 + 4i)|$$

together form a circle in the complex plane and find the centre and radius of the circle.

### Solution

| | | | |
|---|---|---|---|
| Given: | $\lvert z - (8+i)\rvert$ | $=$ | $2\lvert z - (2+4i)\rvert$ |
| Thus if $z = x + iy$ | $\lvert x + iy - (8+i)\rvert$ | $=$ | $2\lvert x + iy - (2+4i)\rvert$ |
| | $(x-8)^2 + (y-1)^2$ | $=$ | $2^2[(x-2)^2 + (y-4)^2]$ |
| | $x^2 - 16x + 64 + y^2 - 2y + 1$ | $=$ | $2^2[x^2 - 4x + 4 + y^2 - 8y + 16]$ |
| Which simplifies to | $0$ | $=$ | $3x^2 + 3y^2 - 30y + 15$ |
| i.e. | $0$ | $=$ | $x^2 + y^2 - 10y + 5$ |
| Create gaps: | $x^2 + y^2 - 10y +$ | $=$ | $-5$ |
| Complete the square | $x^2 + y^2 - 10y + 25$ | $=$ | $-5 + 25$ |
| Hence | $x^2 + (y-5)^2$ | $=$ | $20$ |

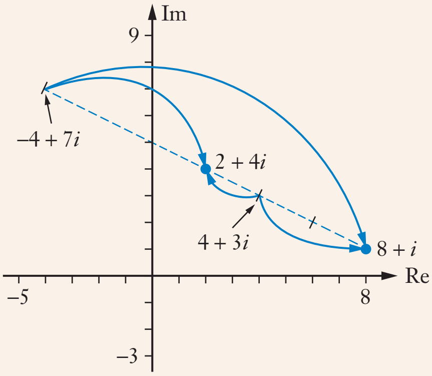
A circle, centre $(0, 5)$ and radius $2\sqrt{5}$.

Note that points in the complex plane such that

$$|z - (8 + i)| = 2|z - (2 + 4i)|$$

are the points for which the distance to the point $(8 + i)$ is twice that of the distance to the point $(2 + 4i)$.

We would expect two such points to be $4 + 3i$ and $-4 + 7i$ (see diagram). The reader should confirm that, for $z = x + iy$, such points do indeed satisfy the equation $x^2 + (y - 5)^2 = 20$.





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## Exercise 2D

For each of the following sets, choose the appropriate diagram from those labelled A to P below.

**1** $\{z:\ \text{Im}(z) = 3\}$

**2** $\{z:\ \text{Re}(z) = 3\}$

**3** $\{z: \arg z = -45°\}$

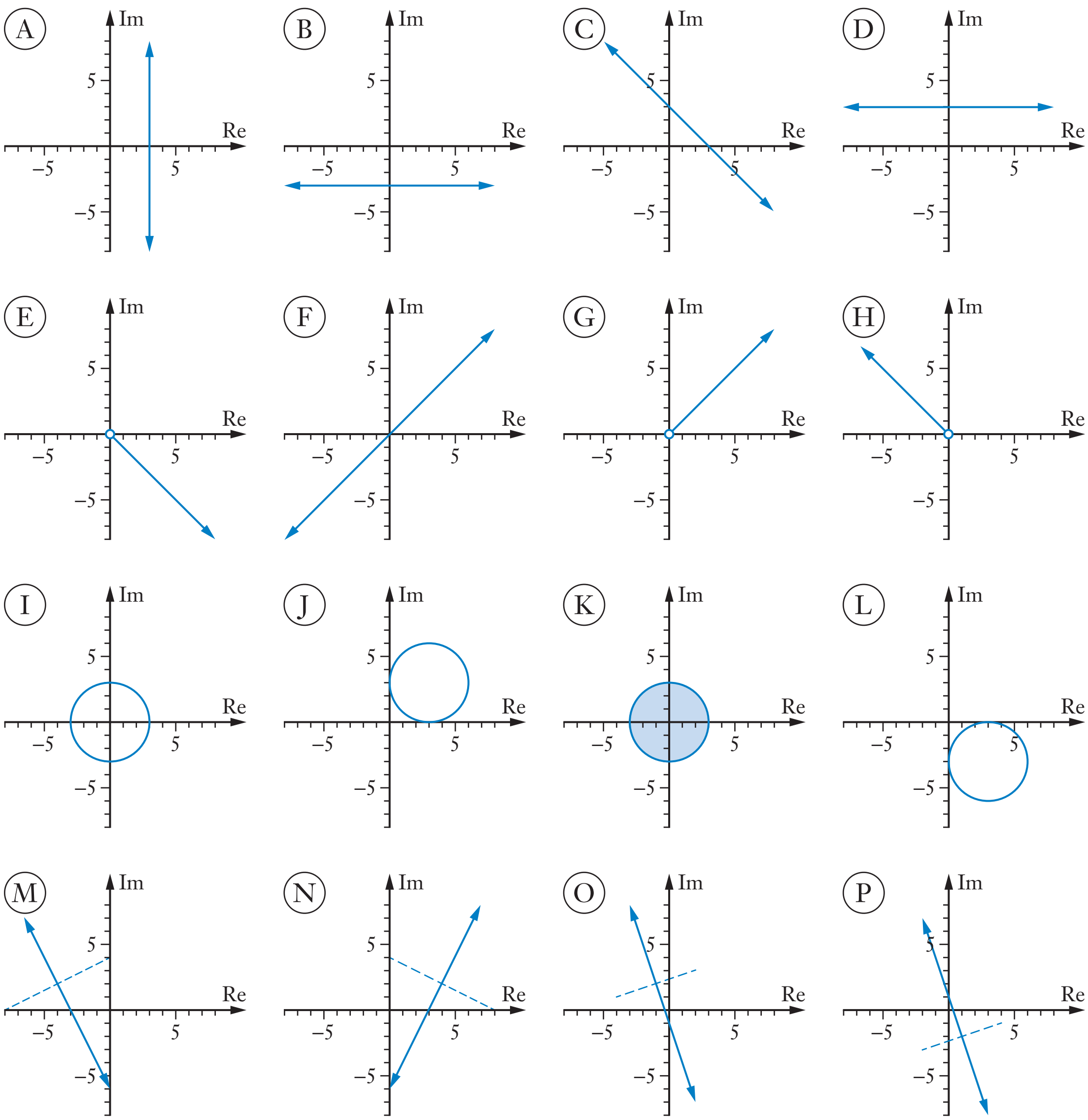
**4** $\{z: \arg z = 135°\}$

**5** $\{z: |z| \le 3\}$

**6** $\{z: |z - 3 + 3i| = 3\}$

**7** $\{z: |z + 8| = |z - 4i|\}$

**8** $\{z: |z + 2 + 3i| = |z - 4 + i|\}$

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Represent each of the following sets of points diagrammatically as lines or regions in the complex plane and, with $z = x + iy$, determine the cartesian equation of each.

**9** $\{z: \text{Re}(z) = 5\}$

**10** $\{z: \text{Im}(z) = -4\}$

**11** $\left\{z: \arg z = \frac{\pi}{3}\right\}$

**12** $\left\{z: \arg z = -\frac{\pi}{3}\right\}$

**13** $\{z: \text{Re}(z) + \text{Im}(z) = 6\}$

**14** $\{z: |z| = 6\}$

**15** $\{z: |z - 4i| \leq 3\}$

**16** $\{z: |z - (2 + 3i)| = 4\}$

**17** $\{z: |z - 2 + 3i| = 4\}$

**18** $\{z: |z - 2| = |z - 6|\}$

**19** $\{z: |z - 6i| = |z - 2|\}$

**20** $\{z: |z - (2 + i)| = |z - (4 - 5i)|\}$

**21** $\{z: 3 \leq |z| \leq 5\}$

**22** $\left\{z: \frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}\right\}$

**23** $\{z: \text{Im}(z) \geq 2\,\text{Re}(z) + 1\}$

**24** $\{z: \text{Im}(z) < 2 - \text{Re}(z)\}$

**25** For $\{z: |z + 3 - 3i| = 2\}$ determine:

- **a** the minimum possible value of $\text{Im}(z)$,
- **b** the maximum possible value of $|\text{Re}(z)|$,
- **c** the minimum possible value of $|z|$,
- **d** the maximum possible value of $|z|$,
- **e** the maximum possible value of $|\overline{z}|$.

**26** For $\{z: |z - (4 + 3i)| = 2\}$ determine:

- **a** the minimum possible value of $\text{Im}(z)$,
- **b** the maximum possible value of $\text{Re}(z)$,
- **c** the maximum possible value of $|z|$,
- **d** the minimum possible value of $|z|$,
- **e** the minimum possible value of $\arg(z)$, giving your answer in radians correct to two decimal places,
- **f** the maximum possible value of $\arg(z)$, giving your answer in radians correct to two decimal places.

**27** Show that the set of all points $z$ in the complex plane that are such that

$$|z - (2 + 3i)| = 2|z - (5 - 3i)|$$

together form a circle in the complex plane and find the centre and radius of the circle.

**28** Show that the set of all points $z$ in the complex plane that are such that

$$|z - (10 + 5i)| = 3|z - (2 - 3i)|$$

together form a circle in the complex plane and find the centre and radius of the circle.


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# The cube roots of 1

Suppose we are asked to solve the equation

$$x^3 = 1,$$

i.e.

$$x^3 - 1 = 0$$

We could use a calculator to obtain the three solutions:

$$x = 1$$

$$x = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

and

$$x = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

solve($x^3 = 1, x$)

$$\left\{x = 1,\ x = -\frac{1}{2} - \frac{\sqrt{3} \cdot i}{2},\ x = -\frac{1}{2} + \frac{\sqrt{3} \cdot i}{2}\right\}$$

Alternatively these three solutions can be obtained algebraically by first using the fact that

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

(The reader should check the truth of this statement by expanding the right hand side.)

Then, if

$$x^3 - 1 = 0$$

$$(x - 1)(x^2 + x + 1) = 0$$

Thus either $x - 1 = 0$ or $x^2 + x + 1 = 0$

giving $x = 1$ or

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

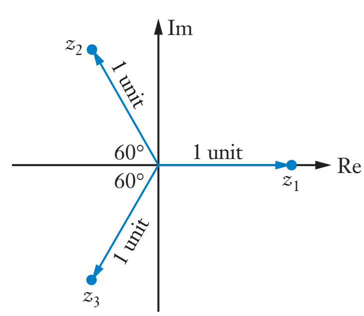
Thus the three cube roots of 1 are: $1$,

$$-\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

and

$$-\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

These three cube roots of 1 are shown as $z_1$, $z_2$ and $z_3$ in the Argand diagram on the right.




In 'cis' form

$$z_1 = 1 \operatorname{cis} 0$$

$$z_2 = 1 \operatorname{cis} \frac{2\pi}{3}$$

$$z_3 = 1 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

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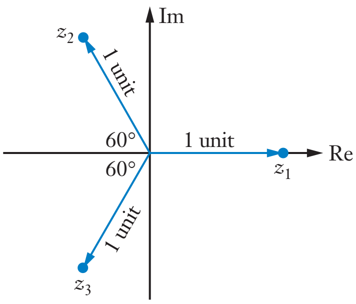
Notice that the three roots are each of unit length and divide the unit circle into three equal size regions. (The reasonableness of this should become apparent when you remember the rotational effect of complex number multiplication.)

This division into equal size regions can be used to determine other roots of 1, as the following example shows.




**EXAMPLE 8**

Find the fifth roots of 1, giving exact answers in the form $r \operatorname{cis} \theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$.

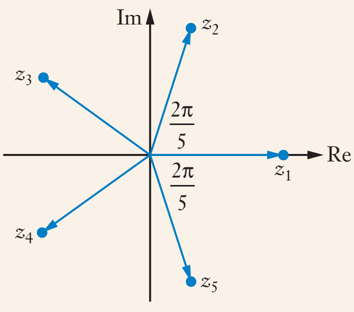
**Solution**

We must solve $\quad z^5 = 1.$

One solution is $\quad z = 1 \quad$ and this and the 4 others will divide the unit circle into five equal-sized regions (see diagram).

Thus the roots are

$$\begin{aligned} z_1 &= 1 \operatorname{cis} 0 \\ z_2 &= 1 \operatorname{cis} \frac{2\pi}{5} \\ z_3 &= 1 \operatorname{cis} \frac{4\pi}{5} \\ z_4 &= 1 \operatorname{cis} \left(-\frac{4\pi}{5}\right) \\ z_5 &= 1 \operatorname{cis} \left(-\frac{2\pi}{5}\right) \end{aligned}$$




## Nth roots of a non-zero complex number

If we know one of the roots of a complex number we can locate all the other roots using this idea of dividing the complex plane into equal size regions. This is demonstrated in the next two examples.

**Note**

Again your calculator may be able to determine the roots of complex numbers directly using its equation solving capabilities or other programmed routines. Whilst you are encouraged to explore the capability of your calculator in this regard make sure you understand the methods set out in the following examples and can use them when required.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## EXAMPLE 9

Use your calculator to confirm that $(1+i)^6$ is $-8i$.

By displaying the sixth roots of $-8i$ on an Argand diagram determine all six roots, expressing each in the form $r\,\text{cis}\,\theta°$ with $r \geq 0$ and $-180 < \theta \leq 180$.

### Solution

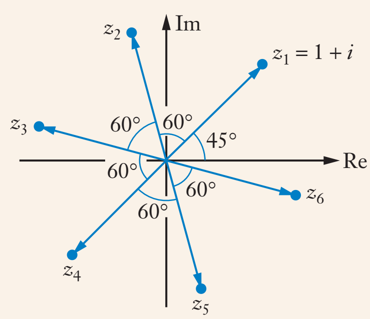
The display shown on the right confirms that $(1+i)^6$ is $-8i$.

$(1+i)^6$

$-8 \cdot i$

Placing $(1 + i)$ on an Argand diagram and dividing the complex plane into six equal size regions allows the six roots to be determined:

$$
\begin{aligned}
z_1 &= \sqrt{2}\,\text{cis}\,45° \\
z_2 &= \sqrt{2}\,\text{cis}\,105° \\
z_3 &= \sqrt{2}\,\text{cis}\,165° \\
z_4 &= \sqrt{2}\,\text{cis}\,(-135°) \\
z_5 &= \sqrt{2}\,\text{cis}\,(-75°) \\
z_6 &= \sqrt{2}\,\text{cis}\,(-15°)
\end{aligned}
$$




## EXAMPLE 10

Use your calculator to confirm that $(3+i)^4$ is $28+96i$. Show $(3 + i)$ and the three other fourth roots of $28 + 96i$ on an Argand diagram and express each in the form $a + bi$.

### Solution

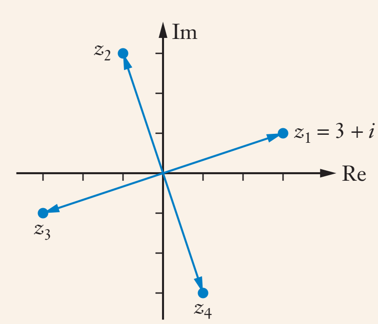
The display shown on the right confirms that $(3+i)^4$ is $28 + 96i$.

$(3+i)^4$

$28 + 96 \cdot i$

Placing $(3 + i)$ on an Argand diagram and dividing the complex plane into four equal size regions allows the four roots to be determined:

$$
\begin{aligned}
z_1 &= 3 + i \\
z_2 &= -1 + 3i \\
z_3 &= -3 - i \\
z_4 &= 1 - 3i
\end{aligned}
$$

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## Exercise 2E

**1** Find the six solutions to the equation $z^6 = 1$ giving exact answers and all in the form $r\,\text{cis}\,\theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$.

**2** Find the eight solutions to the equation $z^8 = 1$ giving exact answers and all in the form $r\,\text{cis}\,\theta°$ with $r \geq 0$ and $-180 < \theta \leq 180$.

**3** Find the seven solutions to the equation $z^7 = 1$ giving exact answers and all in the form $r\,\text{cis}\,\theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$.

**4** Use your calculator to confirm that $(\sqrt{3} + i)^6$ is $-64$. By displaying the sixth roots of $-64$ on an Argand diagram determine all six roots, expressing each root in the form $r\,\text{cis}\,\theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$.

**5** Given that one solution to the equation $z^5 = -4 + 4i$ is $z = 1 - i$, display all five solutions on an Argand diagram and express each in the form $r\,\text{cis}\,\theta°$ with $r \geq 0$ and $-180 < \theta \leq 180$.

**6** Use your calculator to confirm that $(2 + 3i)^4$ is $-119 - 120i$. Show $(2 + 3i)$ and the three other fourth roots of $-119 - 120i$ on an Argand diagram and express each in the form $a + bi$.

**7** Without the assistance of a calculator, express **a** $(2 + i)^2$ and **b** $(2 + i)^4$ in the form $a + bi$.

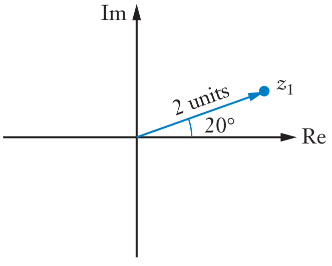
**c** Display on an Argand diagram the four values of $z$ for which

$$z^4 = -7 + 24i$$

**d** Hence give the solutions to the above equation in the form $a + bi$.

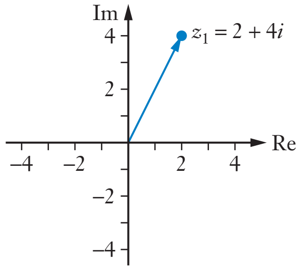
**8** $z_1$ shown in the diagram on the right is one solution to the equation $z^5 = k$, for complex $k$.

Find $k$ and the other four solutions to the equation, giving all answers in the form $r\,\text{cis}\,\theta°$ for $r \geq 0$ and $-180 < \theta \leq 180$.




**9** $z_1$ shown in the diagram on the right is one solution to the equation $z^4 = k$, for complex $k$.

Find $z_2$, $z_3$ and $z_4$, the other three solutions to the equation, giving all answers in the form $a + bi$.

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# De Moivre's theorem

Using de Moivre's theorem

From $(r_1 \operatorname{cis}\theta)(r_2 \operatorname{cis}\alpha) = r_1 r_2 \operatorname{cis}(\theta + \alpha)$

it follows that $\operatorname{cis}\theta \operatorname{cis}\alpha = \operatorname{cis}(\theta + \alpha)$

and hence $(\operatorname{cis}\theta)^2 = \operatorname{cis}(\theta + \theta) = \operatorname{cis}(2\theta)$.

Continuing this idea:

$$
\begin{aligned}
(\operatorname{cis}\theta)^3 &= \operatorname{cis}(\theta + \theta + \theta) &&= \operatorname{cis}(3\theta) \\
(\operatorname{cis}\theta)^4 &= \operatorname{cis}(\theta + \theta + \theta + \theta) &&= \operatorname{cis}(4\theta). \\
&\vdots && \vdots \\
(\operatorname{cis}\theta)^n &= \operatorname{cis}(\theta + \theta + \theta + \ldots) &&= \operatorname{cis}(n\theta).
\end{aligned}
$$

i.e.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

This result is called **de Moivre's theorem** and, whilst we have obtained it by considering positive integer values of $n$, it can be applied for all rational values of $n$.

Thus, with $z = |z| \operatorname{cis}\theta$

it follows that $z^n = |z|^n \operatorname{cis}(n\theta)$

i.e.

$$(|z| \operatorname{cis}\theta)^n = |z|^n \operatorname{cis}(n\theta)$$

an alternative statement of de Moivre's theorem.

In your study of *Mathematics Specialist Unit Two* you encountered the method of *proof by induction*.

(If you have forgotten the technique, do a bit of revision to refresh your understanding of it.)

Use the method of proof by induction to prove that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

i.e. de Moivre's theorem, is true for positive integer values of $n$.

The next three examples show how de Moivre's theorem can be used:

- to obtain expressions for $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$ (example 11)
- to find powers of a complex number (example 12)
- to find the $n$th roots of a complex number (example 13).

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## EXAMPLE 11

Use de Moivre's theorem to express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

### Solution

By de Moivre's theorem

$$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4 \qquad [1]$$

The right hand side of this statement expands to give:

$$(\cos\theta)^4 + 4(\cos\theta)^3(i\sin\theta) + 6(\cos\theta)^2(i\sin\theta)^2 + 4(\cos\theta)(i\sin\theta)^3 + (i\sin\theta)^4$$

$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

Equating real parts of equation [1] gives:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

and equating imaginary parts of equation [1] gives:

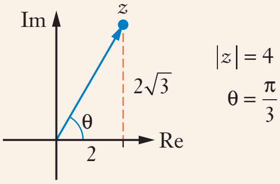
$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

## EXAMPLE 12

Use de Moivre's theorem to determine $(2 + 2\sqrt{3}i)^4$, giving your answer in exact polar form.

### Solution

First change $2 + 2\sqrt{3}i$ to 'cis form'.




$$2 + 2\sqrt{3}i = 4\,\text{cis}\,\frac{\pi}{3}$$

Thus $$(2 + 2\sqrt{3}i)^4 = \left(4\,\text{cis}\,\frac{\pi}{3}\right)^4$$

$$= 4^4\,\text{cis}\,\frac{4\pi}{3} \qquad \text{(from de Moivre's theorem)}$$

$$= 256\,\text{cis}\left(-\frac{2\pi}{3}\right)$$

### Note

The next example uses the fact that whilst we normally write the polar form of a complex number, $z$, as $r\,\text{cis}\,\theta$, with $r \geq 0$ and $-\pi < \theta \leq \pi$, we could write $z$ as $z = r\,\text{cis}\,(\theta + 2k\pi)$, for $r \geq 0$ and $k$ an integer.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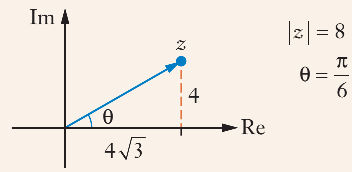
## EXAMPLE 13

Use de Moivre's theorem to determine the three cube roots of $(4\sqrt{3}+4i)$, giving your answer in exact polar form.

### Solution




$$4\sqrt{3}+4i = 8\,\text{cis}\left(\frac{\pi}{6}+2k\pi\right) \quad \text{for integer } k.$$

Thus we require $z$ such that

$$z^3 = 8\,\text{cis}\left(\frac{\pi}{6}+2k\pi\right)$$

$$\therefore \quad z = \left[8\,\text{cis}\left(\frac{\pi}{6}+2k\pi\right)\right]^{\frac{1}{3}}$$

$$= \sqrt[3]{8}\,\text{cis}\left(\frac{\pi}{18}+\frac{2k\pi}{3}\right) \quad \text{(by de Moivre's theorem)}$$

$$= 2\,\text{cis}\left(\frac{\pi}{18}+\frac{2k\pi}{3}\right)$$

If $k=0$ we have: $2\,\text{cis}\left(\frac{\pi}{18}\right)$ (Sometimes referred to as the **principal** root.)

$k=1$ gives: $2\,\text{cis}\left(\frac{13\pi}{18}\right)$

$k=2$ gives: $2\,\text{cis}\left(\frac{25\pi}{18}\right)$ i.e. $2\,\text{cis}\left(-\frac{11\pi}{18}\right)$.

Thus the three cube roots of $4\sqrt{3}+4i$ are $2\,\text{cis}\left(\frac{\pi}{18}\right)$, $2\,\text{cis}\left(\frac{13\pi}{18}\right)$ and $2\,\text{cis}\left(-\frac{11\pi}{18}\right)$.

Wondering what happens if we continue the above process by letting $k = 3, 4$, etc.? Are there more than three cube roots? If you are not sure what will happen try it and see.

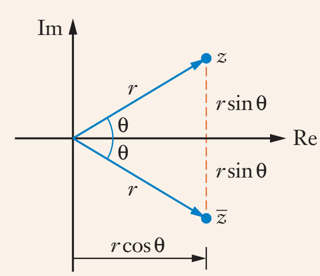
## EXAMPLE 14

The complex number $z$ is such that $z = r\,\text{cis}\,\theta$, with $r > 0$ and $0 \le \theta \le \frac{\pi}{2}$.

Express **a** $\bar{z}$ **b** $-z$ **c** $z^2$

in the form $a\,\text{cis}\,\beta$ where $a > 0$ and $-\pi < \beta \le \pi$.

### Solution




**a** If $z = r\,\text{cis}\,\theta$

$= r\cos\theta + ir\sin\theta$

then $\bar{z} = r\cos\theta - ir\sin\theta$

$= r\,\text{cis}(-\theta)$. (See diagram.)

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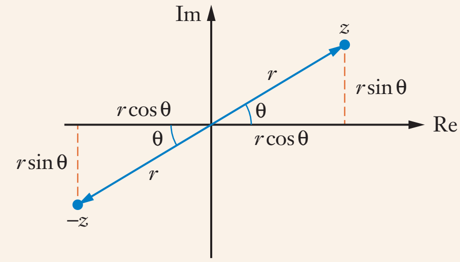
**b** If

$$\begin{aligned} z &= r\,\text{cis}\,\theta \\ &= r\cos\theta + ir\sin\theta \end{aligned}$$

then

$$\begin{aligned} -z &= -r\cos\theta - ir\sin\theta \\ &= r\,\text{cis}\,(-\pi + \theta) \qquad \text{(See diagram.)} \\ &= r\,\text{cis}\,(\theta - \pi) \end{aligned}$$




Or, using the fact that $-1$ can be written as $1\,\text{cis}\,\pi$:

$$\begin{aligned} -z &= -1 \times z \\ &= (1\,\text{cis}\,\pi) \times (r\,\text{cis}\,\theta) \\ &= r\,\text{cis}\,(\pi + \theta) \end{aligned}$$

But $(\pi + \theta)$ will be outside the $-\pi$ to $\pi$ requirement, so subtract $2\pi$ to give

$$\begin{aligned} -z &= r\,\text{cis}\,(\pi + \theta - 2\pi) \\ &= r\,\text{cis}\,(\theta - \pi), \qquad \text{as before.} \end{aligned}$$

**c** If

$$\begin{aligned} z &= r\,\text{cis}\,\theta \\ z^2 &= (r\,\text{cis}\,\theta)^2 \\ &= r^2\,\text{cis}\,(2\theta) \qquad \text{(by de Moivre's theorem).} \end{aligned}$$

## Exercise 2F

**1** Prove that de Moivre's theorem, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, is true for $n = -1$.

**2** If $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ determine $z^4$ in exact polar form.

**3** If $z = 2\,\text{cis}\,\frac{\pi}{6}$ determine $z^5$ in exact polar form.

**4** If $z = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ determine $z^5$ in exact polar form.

**5** Use de Moivre's theorem to express $\cos 2\theta$ and $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$.

**6** Use de Moivre's theorem to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\sin\theta$ and $\cos\theta$. Hence determine $\cos 3\theta$ in terms of $\cos\theta$.

**7** Use de Moivre's theorem to express $\cos 5\theta$ and $\sin 5\theta$ in terms of $\sin\theta$ and $\cos\theta$.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**8** Use de Moivre's theorem to determine $(1 + i)^6$, giving your answer in exact polar form.

**9** Use de Moivre's theorem to determine $(\sqrt{3} + i)^5$, giving your answer in exact polar form.

**10** Use de Moivre's theorem to determine $(-3 + 3\sqrt{3}i)^4$, giving your answer in exact polar form.

**11** Use de Moivre's theorem to determine the three cube roots of $(4 - 4\sqrt{3}i)$, giving your answers in exact polar form.

**12** Use de Moivre's theorem to solve the equation $z^4 = 16i$, giving your answers in exact polar form.

**13** Use de Moivre's theorem to solve the equation $z^4 = -8\sqrt{2} + 8\sqrt{2}i$, giving your answers in exact polar form.

**14** Use de Moivre's theorem to solve the equation $z^4 + 4 = 0$, giving your answers in exact polar form.

**15** Express $z_1$ and $z_2$ in exact polar form, where $z_1 = \dfrac{\sqrt{2} + i\sqrt{6}}{2}$, and $z_2 = \dfrac{\sqrt{6} + i\sqrt{2}}{2}$.

Hence simplify $\dfrac{z_1{}^6 z_2{}^3}{z_3{}^4}$ given that $z_3 = 2\,\text{cis}\,\dfrac{\pi}{8}$.

**16** The complex number $z$ is such that $z = r\,\text{cis}\,\theta$, with $r > 0$ and $0 \le \theta \le \dfrac{\pi}{2}$.

Express each of the following in the form $a\,\text{cis}\,\beta$ where $a > 0$ and $-\pi < \beta \le \pi$.

**a** $-\bar{z}$

**b** $\dfrac{1}{z}$

**c** $-\dfrac{1}{z}$

**d** $-\dfrac{1}{z^2}$


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Abraham de Moivre (1667–1754)

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# Miscellaneous exercise two

**This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.**

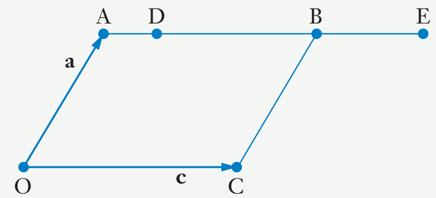
**1** If the complex numbers $z$ and $w$ are such that $z = 3 - 4i$ and $w = 2 + 3i$

express each of the following in the form $a + bi$.

**a** $z + w$ **b** $z - w$ **c** $zw$

**d** $z^2$ **e** $\frac{z}{w}$ **f** $\frac{w}{z}$

**2** The diagram on the right shows the parallelogram OABC with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.




D is a point on AB such that AD : DB = 1 : 3.

E is a point on AB produced such that AB : BE = 2 : 1.

Express each of the following in terms of **a** and/or **c**.

**a** $\overrightarrow{AB}$ **b** $\overrightarrow{AD}$ **c** $\overrightarrow{DB}$ **d** $\overrightarrow{DE}$

**e** $\overrightarrow{OB}$ **f** $\overrightarrow{OD}$ **g** $\overrightarrow{CE}$ **h** $\overrightarrow{OE}$

**3** Express exactly

**a** $-3 - 3\sqrt{3}i$ in polar form,

**b** $8\,\text{cis}\left(-\frac{5\pi}{6}\right)$ in cartesian form.

**4** Express each of the following in the form $(a, b)$ where $(a, b)$ represents the complex number $a + bi$.

**a** $2\,\text{cis}\left(\frac{\pi}{2}\right)$ **b** $5\,\text{cis}\,\pi$ **c** $4\,\text{cis}\left(-\frac{3\pi}{4}\right)$

**5** If $z = 1 + i$ and $w = -1 + i$ express $z$, $w$, $zw$ and $\frac{z}{w}$ in polar form.

**6** With $\text{cis}\,\theta$ defined as $\cos\theta + i\sin\theta$, where $i = \sqrt{-1}$, prove that:

**a** $\text{cis}\,0 = 1$ **b** $\text{cis}\,\alpha\,\text{cis}\,\beta = \text{cis}\,(\alpha + \beta)$

**7** (Without the assistance of your calculator.)

**a** For $f(x) = 4x^3 - 18x^2 + 22x - 12$, determine $f(-3)$ and $f(3)$.

**b** Determine all values for $x$, real and complex, for which $f(x) = 0$.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